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THE ROLLING OF A WHEEL WITH A REINFORCED TYRE ALONG A PLANE WITH SLIPPING†

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A previously proposed model [1] of a wheel with a reinforced tyre in which the side walls of the tyre are represented by reinforced membranes consisting of incompressible rubber, in accordance with the Mooney–Rivlin model [2], is considered. When the tread is deformed, the exact non-linear conditions for the inextensibility of the reinforcing cords are taken into account, unlike the conditions in the linearized form used previously in [1]. A potential-energy functional of the deformed tyre as a function of the deformations of the centre line of the tread and the displacements of the wheel disc, which has six degrees of freedom, is obtained using a number of hypotheses. A complete system of equations is derived and the conditions imposed on the sudden jumps in the functions describing the deformation of the tread at the boundary points of the previously unknown area of contact of the tread with the plane when there is slipping is obtained using a model of dry friction. Two steady modes of motion of the locked wheel are investigated: rectilinear translational motion at a constant speed and spinning around an axis orthogonal to the rolling plane of the wheel with constant angular velocity. © 2005 Elsevier Ltd. All rights reserved.

A criterion for the transition from the mode in which the wheel is spinning and slipping to the mode in which spinning occurs without slipping was proposed in [3]. The dynamic interaction between deformable rigid bodies was investigated in [4–7] using a model of dry friction.

1. MODELLING OF A WHEEL WITH A REINFORCED TYRE

We will assume that the wheel with the reinforced tyre consists of a disc (0) (a rigid body), joined to the side wall of the tyre (1, 2), which is represented by the parts of two tori, and a tread (3), reinforced with inextensible steel cords (Fig. 1). We will introduce a fixed system of coordinates $OX_1X_2X_3$ (the wheel is in contact with the surface OX_1X_2) and a moving system of coordinates $Cx_1x_2x_3$ with origin at the mass centre of the disc at the point C. The radius vector of a point on the tread is defined in the form

$$\mathbf{R}_3(\varphi, \xi, t) = \sum_{i=1}^{3} X_i\mathbf{l}_i + r\Gamma_3(\beta_0)\Gamma_2(\theta+\varphi)\left[\mathbf{e}_1 + lr^{-1}\xi\mathbf{e}_2 + \sum_{i=1}^{3} U_i(\varphi, \xi, t)\mathbf{e}_i\right]$$

$$\Gamma_2(\theta) = \begin{Vmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{Vmatrix}, \quad \Gamma_3(\beta) = \begin{Vmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad \underset{|\xi|\le 1}{\varphi\,\mathrm{mod}\,2\pi} \tag{1.1}$$

where X_1, X_2 and X_3 are the coordinates of the mass centre of the disc. $\mathbf{l}_i$ is a unit vector of the axis OX_i, β_0 and θ are the angles of rotation about the axes OX_3 and Cx_2 respectively, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are unit vectors of a cylindrical system of coordinates, r and $2l$ are the radius and width of the tread, and U_1, U_2 and U_3 are the components of the displacement vector of a point on the tread in a cylindrical system

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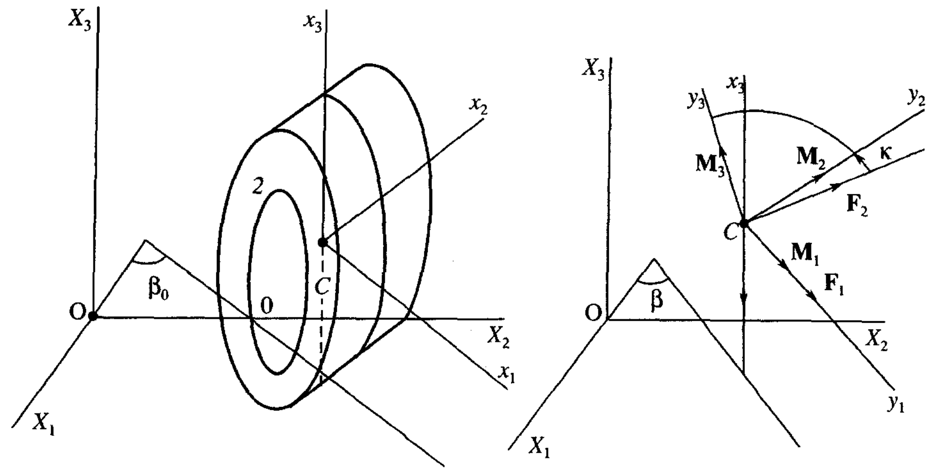

Fig. 1

of coordinates. The tread consists of a rubber strip, reinforced by steel cords, for which the conditions for inextensibility, represented by the equations

$$\left|\frac{\partial \mathbf{R}_3}{l\partial\xi}\right| = 1, \quad \left|\cos\gamma_0\frac{\partial \mathbf{R}_3}{l\partial\xi} \pm \sin\gamma_0\frac{\partial \mathbf{R}_3}{r\partial\varphi}\right| = 1, \quad \gamma_0 = \text{const}$$

and the equivalent equalities

$$\begin{aligned}
&|l^{-1}\mathbf{R}_3^\circ| = 1 \Rightarrow 2lr^{-1}U_2^\circ + \sum_{k=1}^{3} U_k^{\circ 2} = 0; \quad (\cdot)^\circ = \frac{\partial(\cdot)}{\partial\xi} \\
&|r^{-1}\mathbf{R}_3'| = 1 \Rightarrow 2(U_1 - U_3') + (U_1 - U_3')^2 + U_2'^2 + (U_1' + U_3)^2 = 0 \\
&\mathbf{R}_3^\circ \cdot \mathbf{R}_3' = 0 \Rightarrow U_1^\circ(U_1' + U_3) + (lr^{-1} + U_2^\circ)U_2' - U_3^\circ(1 + U_1 - U_3') = 0; \quad (\cdot)' = \frac{\partial(\cdot)}{\partial\varphi}
\end{aligned} \tag{1.2}$$

are satisfied.

We previously [1] considered linearized relations (1.2), from which we obtained formulae expressing U_1, U_2 and U_3 as functions of the displacements of the centre line of the tread $u(\varphi, t)$, $\upsilon(\varphi, t)$ and $w(\varphi, t)$, namely

$$U_1 = lr^{-1}\xi w'' + u, \quad U_2 = w, \quad U_3 = lr^{-1}\xi w' - \upsilon \tag{1.3}$$

If relations (1.3) are substituted into the exact conditions of inextensibility and orthogonality of the filaments of the cord, it turns out that the function w depends only on the time and $w' = w'' = 0$, while the functions u and υ satisfy the condition that the middle cord of the tread is inextensible

$$2(u + \upsilon') + (u + \upsilon')^2 + (\upsilon - u')^2 = 0 \tag{1.4}$$

Hence, equalities (1.3) take the form

$$U_1 = u(\varphi, t), \quad U_2 = w(t), \quad U_3 = -\upsilon(\varphi, t) \tag{1.5}$$

The result obtained expresses the fact that the surface of the circular cylinder (the undeformed tread) when the cord is inextensible and orthogonal, corresponding to a change in the coordinates ξ and φ, is isometric with a cylindrical surface with generatrix specified by the deformed plane centre line of the tread, and by a family of straight lines orthogonal to it. Since the tread of the tyre is in contact with the OX_1X_2 plane, this family of straight lines is parallel to the OX_1X_2 plane and makes and angle $\beta_0 + \pi/2$ with the OX_1 axis. In the contact area $L_1 = [\varphi_1(t), \varphi_2(t)]$ the holonomic relation

$$\mathbf{R}_3(\varphi, \xi, t) \cdot \mathbf{l}_3 = 0 \Rightarrow r^{-1}X_3 - (1 + u)\sin\vartheta - \upsilon\cos\vartheta = 0, \quad \vartheta = \theta + \varphi \tag{1.6}$$

holds.

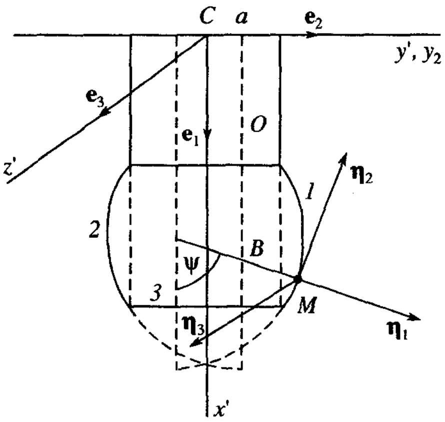

Fig. 2

Conditions (1.4) and (1.6) are equivalent to the following equations

$$u = \left(\vartheta - \frac{\pi}{2}\right)\cos\vartheta + \frac{X_3}{r}\sin\vartheta - 1, \quad \upsilon = -\left(\vartheta - \frac{\pi}{2}\right)\sin\vartheta + \frac{X_3}{r}\cos\vartheta \tag{1.7}$$

The position of the edge points of the tread is determined by vector field (1.1) when $\xi = (-1)^{j+1}$ $(j = 1, 2)$

$$\mathbf{R}_3(\varphi, (-1)^{j+1}, t) = \sum_{i=1}^{3} X_i \mathbf{l}_i + r\Gamma_3(\beta_0)\Gamma_2(\vartheta)[(1+u)\mathbf{e}_1 + ((-1)^{j+1} l r^{-1} + w)\mathbf{e}_2 - \upsilon \mathbf{e}_3] \tag{1.8}$$

We will connect with the wheel disc a system of coordinates $Cy_1y_2y_3$ and define the radius vectors of points of the side walls of the tyre by the relations [1]

$$\mathbf{R}_j(\varphi, \psi, t) = \sum_{i=1}^{3} X_i \mathbf{l}_i + \Gamma_3(\beta)\Gamma_1(\kappa)\Gamma_2(\vartheta)\left\{(-1)^j a\mathbf{e}_2 + c\mathbf{e}_1 + b\Gamma_3(\psi)\left[\boldsymbol{\eta}_1 + \sum_{i=1}^{3} V_i \boldsymbol{\eta}_i\right]\right\}$$

$$\psi \in I_j, \quad I_j = [(-1)^{j+1}\psi_j, (-1)^{j+1}\psi_{3-j}], \quad j = 1, 2, \quad \Gamma_1(\kappa) = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & \cos\kappa & -\sin\kappa \\ 0 & \sin\kappa & \cos\kappa \end{matrix} \right\| \tag{1.9}$$

where η_1, η_2, η_3 are the unit vectors of a toroidal system of coordinates, and a, b and c are constant quantities (Figs 1 and 2). The angles of rotation β and κ of the system of coordinates connected with the disc and the angle β_0 in relations (1.1) are such that the quantities $\beta-\beta_0$ and κ are small, since they define the displacement of the tyre tread with respect to the disc due to deformations of the side walls of the tyre. We will assume that the sidewalls of the tyre are a membrane consisting of incompressible rubber (the Mooney–Rivlin model [2]), reinforced by inextensible steel cords, corresponding to a change in the angle ψ, i.e.

$$\left|\frac{\partial \mathbf{R}_j}{b\partial\psi}\right| = 1 \Rightarrow 2(V_1^{\cdot} + V_1) + (V_2^{\cdot} + V_1)^2 + (V_1^{\cdot} - V_2)^2 + V_3^{\cdot 2} = 0, \quad (\cdot)^{\cdot} = \frac{\partial(\cdot)}{\partial\psi}$$

Apart from terms of the first order of smallness in the components of the vector $\mathbf{V}$ and their derivatives, the last equation can be represented in the form

$$V_2^{\cdot} + V_1 = 0 \tag{1.10}$$

The edges of the side wall of the tyre are joined to the disc of the tyre and to the edges of the tread, whence we have the equations

$$\mathbf{V}(\varphi, (-1)^{j+1}\psi_2, t) = 0, \quad \mathbf{R}_3(\varphi, (-1)^{j+1}, t) = \mathbf{R}_j(\varphi, (-1)^{j+1}\psi_1, t), \quad j = 1, 2 \tag{1.11}$$

Taking relations (1.1) and (1.8) into account, the last two equations have the form

$$r\Gamma_3(\beta_0)\Gamma_2(\vartheta)\{(1+u)\mathbf{e}_1+[(-1)^{j+1}lr^{-1}+w]\mathbf{e}_2-\upsilon\mathbf{e}_3\} = \\ = \Gamma_3(\beta)\Gamma_1(\kappa)\Gamma_2(\vartheta)\left[(-1)^j a\mathbf{e}_2+c\mathbf{e}_1+b\Gamma_3((-1)^{j+1}\psi_1)\left(\boldsymbol{\eta}_1+\sum_{i=1}^{3}V_i\boldsymbol{\eta}_i\right)\right] \tag{1.12}$$

In the undeformed state, the edge of the tread and side walls coincide and the following equality holds

$$r\mathbf{e}_1+(-1)^{j+1}l\mathbf{e}_2 = (-1)^j a\mathbf{e}_2+c\mathbf{e}_1+b\Gamma_3((-1)^{j+1}\psi_1)\boldsymbol{\eta}_1$$

taking account of which, we obtain from relation (1.12) the equations

$$\begin{aligned} bV_1(\varphi,(-1)^{j+1}\psi_1,t) &= (-1)^{j+1}r[w-(1+u)(\Delta\beta\cos\vartheta+\kappa\sin\vartheta)+ \\ &+\upsilon(\Delta\beta\sin\vartheta-\kappa\cos\vartheta)]-\frac{l}{2}[(\Delta\beta)^2+\kappa^2] \\ bV_2(\varphi,(-1)^{j+1}\psi_1,t) &= (-1)^j r\left[u-\frac{1}{2}(\Delta\beta\cos\vartheta+\kappa\sin\vartheta)^2\right]+ \\ &+[(-1)^j rw-l](\Delta\beta\cos\vartheta+\kappa\sin\vartheta), \quad j=1,2 \\ bV_3(\varphi,(-1)^{j+1}\psi_1,t) &= (-1)^{j+1}l(\Delta\beta\sin\vartheta-\kappa\cos\vartheta)-r\upsilon, \quad \Delta\beta=\beta-\beta_0 \end{aligned} \tag{1.13}$$

In deriving Eqs (1.13) we took into account the constructional features of the tyre, when the angle ψ_1 is close to $\pi/2$, and we correspondingly took $\cos\psi_1 \approx 0$ and $\sin\psi_1 \approx 1$. In (1.13) we have retained terms of the second order of smallness in the first and second relations, while the third relation contains only linear terms in the small quantities u, υ, w, $\Delta\beta$ and κ. It is necessary to take into account terms of the second order of smallness when calculating the effect of pressure on possible displacements.

The potential energy of the stretching of the rubber in the Mooney–Rivlin model is represented by the functional

$$\mathrm{E}[\mathbf{V}] = b^2\int_0^{2\pi}\int_{I_1\cup I_2}[k_1(\mathrm{I}_c-3)+k_2(\mathrm{II}_c-3)]\left(\frac{c}{b}+\cos\psi\right)d\varphi d\psi, \quad \mathrm{III}_c = 1 \tag{1.14}$$

where k_1 and k_2 are positive constant coefficients, and I_c, II_c and III_c are the invariants of the tensor of the finite Cauchy–Green deformations [2]. In the case of a two-dimensional continuous medium the Cauchy–Green tensor C_2 is given by the relations

$$d\mathbf{R}_j^2 = \left(\frac{\partial\mathbf{R}_j}{\partial\varphi}\right)^2 d\varphi^2+2\frac{\partial\mathbf{R}_j}{\partial\varphi}\frac{\partial\mathbf{R}_j}{\partial\psi}d\varphi d\psi+b^2d\psi^2 = \mathbf{C}_2\mathbf{B}\mathbf{B}$$

$$\mathbf{C}_2 = \begin{Vmatrix} c_{11} & c_{12} \\ c_{12} & 1 \end{Vmatrix}, \quad \mathbf{B} = \begin{Vmatrix} (c+b\cos\psi)d\varphi \\ bd\psi \end{Vmatrix}$$

The principal elongations λ_1 and λ_2 in this case satisfy the equations

$$\lambda_1^2+\lambda_2^2 = c_{11}+1, \quad \lambda_1^2\lambda_2^2 = c_{11}-c_{12}^2$$

After calculations we obtain

$$c_{11} = 1+Y_1+Y_2, \quad Y_1 = -\frac{2b}{c+b\cos\psi}[V_3'+V_2\sin\psi-V_1\cos\psi]$$

$$c_{12} = (V_2'-V_3\sin\psi)\frac{b}{c+b\cos\psi}-V_3^{\boldsymbol{\cdot}}$$

where Y_2 is a small second-order quantity in the components of the vector **V** and their derivatives. In the three-dimensional case, the Cauchy–Green tensor is equal to $\mathbf{C} = (\partial\mathbf{R}/\partial\mathbf{r})^T(\partial\mathbf{R}/\partial\mathbf{r})$, $\mathbf{R} = \mathbf{R}(\mathbf{r}, t)$, and its invariants are related to the principal elongations λ_1, λ_2 and λ_3 as follows:

$$\mathrm{I}_c = \sum_{k=1}^{3} \lambda_k^2, \quad \mathrm{II}_c = \sum_{i<j}^{3} \lambda_i^2\lambda_j^2, \quad \mathrm{III}_c = \lambda_1^2\lambda_2^2\lambda_3^2$$

In the Mooney–Rivlin incompressible rubber model

$$\mathrm{I}_c = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2}, \quad \mathrm{II}_c = \lambda_1^2\lambda_2^2 + \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2\lambda_2^2}, \quad \mathrm{III}_c = 1$$

Hence, the functional of the potential energy of the deformations (1.14) takes the form

$$\mathrm{E}[\mathbf{V}] = \int_0^{2\pi} \int_{I_1 \cup I_2} \frac{(k_1 + k_2)b^3}{c + b\cos\psi}\Big[4(V_3' - V_1\cos\psi + V_2\sin\psi)^2 +$$

$$+\left(V_2' - V_3\sin\psi - \left(\frac{c}{b} + \cos\psi\right)\dot{V}_3\right)^2\Big]d\varphi d\psi$$

In region I_1 the angle ψ is close to $\pi/2$, while in region I_2 it is close to $-\pi/2$, and correspondingly $\cos\psi \approx 0$ and $\sin\psi \approx 1$ in I_1 and $\sin\psi \approx -1$ in I_2. These approximations will be used when calculating the potential energy of the deformations of the rubber and the work of the pressure in possible displacements. As a result, the functional of the potential energy of deformations of the side walls can be represented by the expression

$$\mathrm{E}[\mathbf{V}] = \int_0^{2\pi} \sum_{j=1}^{2} \int_{I_j} k\Big[4(V_3' - (-1)^j V_2)^2 + \left(V_2' + (-1)^j V_3 - \frac{c}{b}\dot{V}_3\right)^2\Big]d\varphi d\psi, \quad k = \frac{(k_1 + k_2)b^3}{c} \tag{1.15}$$

When evaluating the integrals in (1.15) one must only use the linear terms in the small quantities in expressions (1.13), since the integrand in (1.15) is quadratic in the small components of the displacement vector.

We will obtain the work done by the pressure making virtual displacements when the side walls of the tyre and the tread are deformed. We have

$$\delta A = \sum_{k=1}^{3} \delta A_k, \quad \delta A_3 = p\int_{-1}^{1}\int_0^{2\pi} [\mathbf{R}_3' \times \mathbf{R}_3^\circ]\delta\mathbf{R}_3 d\xi d\varphi$$

$$\delta A_k = p\int_{I_k}\int_0^{2\pi} [\mathbf{R}_k' \times \dot{\mathbf{R}}_k]\delta\mathbf{R}_k d\psi d\varphi, \quad k = 1, 2 \tag{1.16}$$

Here p is the pressure in the tyre, which, as was shown previously in [8], can be assumed to be constant when calculating its work in (1.16), apart from terms of the second order of smallness inclusive. For the tread we obtain from (1.16)

$$\delta A_3 = plr^2\int_{-1}^{1}\int_0^{2\pi} [\delta U_1 - rl^{-1}U_1^\circ\delta U_2 + (U_1' + U_3)\delta U_3]d\xi d\varphi$$

and further, taking relations (1.14) and (1.5) into account, we obtain

$$\delta A_3 = -2plr^2\int_0^{2\pi} (u' - \upsilon)\delta u' d\varphi$$

By relations (1.16) the work done by the pressure in deformations of the side walls, apart from terms of the second order of smallness inclusive, in terms of the functions V_1, V_2 and V_3 and their derivatives, is equal to

$$\delta A_k = pb^3\int_0^{2\pi}\int_{I_k}\Big[\delta V_1\Big(\frac{c}{b}+\cos\psi - V_3' + V_1\cos\psi - V_2\sin\psi\Big) - \delta V_2\Big(\frac{c}{b}+\cos\psi\Big)(V_1^{\cdot} - V_2) +$$

$$+\delta V_3(V_1' + V_3\cos\psi)\Big]d\varphi d\psi, \quad k = 1, 2$$

We will represent the function $V_3(\varphi, \psi, t)$ in the same way as previously [1], taking into account relations (1.13) in the form of the first two terms of an expansion in a Taylor series

$$V_3(\varphi, \psi, t) = b^{-1}[(-1)^k rv + l(\Delta\beta\sin\vartheta - \kappa\cos\theta)](\psi + (-1)^k\psi_2)(\psi_1 - \psi_2)^{-1},$$

$$\psi \in I_k, \quad k = 1, 2$$

Taking relations (1.10) and (1.13) into account, we can also represent the functions V_1 and V_2 in the form of limited expansions in Taylor series in the neighbourhoods of the points $\psi = \pm\psi_2$

$$V_1 = -2(\psi + (-1)^j\psi_2)P_1 + 3(\psi + (-1)^j\psi_2)^2P_2, \quad V_2 = (\psi + (-1)^j\psi_2)^2P_1 - (\psi + (-1)^j\psi_2)^3P_2$$

$$P_1 = -(-1)^j(\psi_1 - \psi_2)^{-1}V_1(\varphi, -(-1)^j\psi_1, t) + 3(\psi_1 - \psi_2)^{-2}V_2(\varphi, -(-1)^j\psi_1, t);$$

$$j = 1, \quad \psi \in I_1$$

$$P_2 = (\psi_1 - \psi_2)^{-2}V_1(\varphi, -(-1)^j\psi_1, t) - 2(-1)^j(\psi_1 - \psi_2)^{-3}V_2(\varphi, -(-1)^j\psi_1, t);$$

$$j = 2, \quad \psi \in I_2$$

As a result, the work done by the pressure and the variation of the potential energy when the side walls and the tread are deformed can be represented in the form

$$\delta A - \delta \mathrm{E}[\mathbf{V}] = -\int_0^{2\pi}\Big[n_0\delta u + \frac{1}{2}n_{01}\delta u^2 + \frac{1}{2}n_{11}\delta u'^2 + \frac{1}{2}n_{02}\delta v^2 + \frac{1}{2}n_{12}\delta v'^2 +$$

$$+ m_{12}u'\delta v + m_{21}v'\delta u + m_{02}u\delta v' + m_{20}v\delta u'\Big]d\varphi - \frac{r^2}{2}n_{03}\delta w^2 - \frac{1}{2}n_{05}\delta(\kappa^2 + \Delta\beta^2) \tag{1.17}$$

The coefficients in Eq. (1.17) are found by evaluating definite integrals (by integrating over ψ, taking into account the evenness and oddness of the corresponding functions). The results of these calculations are not given here in view of their complexity. We will merely note that, taking relation (1.4) into account, expression (1.17) is negative-definite.

The area of contact of the tyre with the OX_1X_2 plane can be represented by a rectangle of width $2l$ and length $r(\varphi_2(t) - \varphi_1(t))$. We will introduce a force field, which describes the interaction between the tyre and the plane, in the contact area. Since the tread of the tyre in the contact area is represented by a flexible flat plate, on which the pressure acts on the inside, it is natural to assume that the normal reaction (the component along the OX_3 axis), describing the interaction of the tyre with the plane, is equal to the pressure p. We will project the velocity field of points of the tyre in the contact area, equal to $\mathbf{R}_3$ (φ, ξ, t) by relation (1.1), onto the axis of the system of coordinates $Ox_1x_2x_3$, obtained by rotating the system of coordinates $OX_1X_2X_3$ by an angle β_0 around the OX_3 axis. Its projection onto the Ox_3 axis is equal to zero, while the projections onto the Ox_1 and Ox_2 axes can be represented by the expressions

$$\dot{Z}_1 = \dot{X}_1\cos\beta_0 + \dot{X}_2\sin\beta_0 - r\dot{\beta}_0(w + lr^{-1}\xi) -$$

$$-r\dot{\theta}[(1+u)\sin\vartheta + v\cos\vartheta] + r(\dot{u}\cos\vartheta - \dot{v}\sin\vartheta) \tag{1.18}$$

$$\dot{Z}_2 = -\dot{X}_1\sin\beta_0 + \dot{X}_2\cos\beta_0 + r\dot{\beta}_0[(1+u)\cos\vartheta - v\sin\vartheta] + r\dot{w}$$

The work done by the forces of Coulomb friction, which act on points of the contact area, making virtual displacement, is given in the form

$$\delta A_f = -fplr\int_{L_1}\int_{-1}^{1}[\dot{Z}_1\delta Z_1+\dot{Z}_2\delta Z_2][\dot{Z}_1^2+\dot{Z}_2^2]^{-1/2}d\varphi d\xi \tag{1.19}$$

Here f is the friction coefficient, while in expressions δZ_1 and δZ_2 the velocities are replaced by the corresponding variations.

We will represent relation (1.19) in the form

$$\delta A_f = -fplr\int_{L_1}\sum_{i=1}^{7}\frac{\partial}{\partial\dot{q}_i}\int_{-1}^{1}\dot{Z}(\varphi,\xi,\dot{\mathbf{q}},\mathbf{q})\delta q_i d\xi d\varphi$$

$$\dot{Z} = (\dot{Z}_1^2+\dot{Z}_2^2)^{1/2},\quad \mathbf{q} = (X_1, X_2, \beta_0, \theta, u, v, w)$$

and further

$$\delta A_f = -\int_{L_1}\sum_{i=1}^{7}\frac{\partial W(\varphi,\dot{\mathbf{q}},\mathbf{q})}{\partial\dot{q}_i}\delta q_i d\varphi \tag{1.20}$$

$$W(\varphi,\dot{\mathbf{q}},\mathbf{q}) = fplr\int_{-1}^{1}\dot{Z}(\varphi,\xi,\dot{\mathbf{q}},\mathbf{q})d\xi = \frac{pfr}{\dot{\beta}_0}\left\{\dot{Z}_1(1)\dot{Z}(1)-\dot{Z}_1(-1)\dot{Z}(-1)+\dot{Z}_2^2\ln\frac{\dot{Z}_1(1)+\dot{Z}(1)}{\dot{Z}_1(-1)+\dot{Z}(-1)}\right\}$$

$$\dot{Z}_1(\pm1) = \dot{Z}_1(\varphi,\pm1,\dot{\mathbf{q}},\mathbf{q}),\quad \dot{Z}(\pm1) = \dot{Z}(\varphi,\pm1,\dot{\mathbf{q}},\mathbf{q})$$

Relation (1.20) holds when $\dot{\beta}_0 \neq 0$. Otherwise the function W does not exist, since the expression in braces and $\dot{\beta}_0$ in the denominator of the fraction vanish, and we must take

$$W(\varphi,\dot{\mathbf{q}},\mathbf{q}) = 2fplr\dot{Z}\big|_{\dot{\beta}_0=0} \tag{1.21}$$

In this case the contact area performs translational motion.

2. THE EQUATIONS OF MOTION

The kinetic energy of the wheel is given by the expression

$$2T = m_d\sum_{i=1}^{3}\dot{X}_i^2+J_{1d}(\dot{\kappa}^2+\dot{\beta}^2\cos^2\kappa)+J_{2d}(\dot{\theta}+\dot{\beta}\sin\kappa)^2+\rho r\int_0^{2\pi}\sum_{i=1}^{3}\dot{Z}_i^2 d\varphi \tag{2.1}$$

where m_d, J_{1d} and J_{2d} are the mass and moments of inertia of the disc about the axes Cy_1 and Cy_2 respectively. The kinetic energy of the tread and the side walls in expression (2.1) is represented by the last term, on the assumption that the whole mass of the tyre is distributed uniformly about the centre line. The quantities $\dot{Z}_1$, $\dot{Z}_2$ are defined in (1.18), and

$$\dot{Z}_3 = \dot{X}_3-r\dot{\theta}[(1+u)\cos\vartheta-v\sin\vartheta]-r(\dot{u}\sin\vartheta+\dot{v}\cos\vartheta) \tag{2.2}$$

The equations of motion and the conditions on the boundary of the contact area, unknown in advance, are obtained from the Hamilton–Ostrogradskii variational principle. To do this we will obtain expressions for the work done by the external forces and moments applied to the wheel disc (Fig. 1), making virtual displacements, namely

$$\delta A_F = F(\beta)\delta X_1+F(\beta-\pi/2)\delta X_2-P\delta X_3+M_1\delta\kappa+M_2\delta\theta+(M_2\sin\kappa+M_3\cos\kappa)\delta\beta$$

$$F(\beta) = F_1\cos\beta-F_2\sin\beta$$

We will use as the constraints for points on the centre line of the tread l_0 the condition for it to be inextensible (1.4), represented in the form.

$$2Z_0 = (1+u+\upsilon')^2 + (u'-\upsilon)^2 = 1 \tag{2.3}$$

Correspondingly, when releasing from these constraints, we must bear in mind their work done on virtual displacements

$$\delta N_0 = \int_0^{2\pi} \lambda(\varphi, t)\delta Z_0 d\varphi$$

where λ is an undetermined Lagrange multiplier.

We will represent the Hamilton–Ostrogradskii variational principle in the form

$$\int_{t_1}^{t_2} \left[\delta T + \delta A_F + \delta A - \delta \mathrm{E} + \delta A_f + \delta N_0 + \int_{L_1} \mu_3(\varphi, t)\delta Z_3 d\varphi\right] dt = 0 \tag{2.4}$$

Variation δZ_3 corresponds to holonomic constraint (1.6) or (2.2), while the factor $\mu_3(\varphi, t)$ is the normal component of the reaction of the constraints, reduced to unit length of the tread in the contact area. The corresponding variables in Eq. (2.4) are 2π-periodic in the variable φ, and the region of integration $[t_1, t_2] \cup [\varphi_1, \varphi_1 + 2\pi]$ in relation (2.24) is divided by the curve $\varphi = \varphi_2(t)$ into two parts, $[t_1, t_2] \cup L_1$ and $[t_1, t_2] \cup L_2$ $(L_2 = [\varphi_2(t), 2\pi + \varphi_1(t)]$. We apply Green's formula to each of these and obtain a system of equations and conditions imposed on the jump at the boundary points of the contact area

$$-\frac{d}{dt}\nabla_{\dot X_1}T - \int_{L_1}\nabla_{\dot X_1}W d\varphi + F(\beta) = 0, \quad -\frac{d}{dt}\nabla_{\dot X_2}T - \int_{L_1}\nabla_{\dot X_2}W d\varphi + F\left(\beta - \frac{\pi}{2}\right) = 0$$

$$-\frac{d}{dt}\nabla_{\dot X_3}T - P + \int_{L_1}\mu_3(\varphi, t)d\varphi = 0$$

$$-\frac{d}{dt}\nabla_{\dot\beta}T + M_2\sin\kappa + M_3\cos\kappa - n_{05}\Delta\beta = 0$$

$$-\frac{d}{dt}\nabla_{\dot\beta_0}T + n_{05}\Delta\beta - \int_{L_1}\nabla_{\dot\beta_0}W d\varphi = 0, \quad \nabla_\kappa T - \frac{d}{dt}\nabla_{\dot\kappa}T + M_1 - n_{05}\kappa = 0$$

$$\nabla_\theta T - \frac{d}{dt}\nabla_{\dot\theta}T + M_2 - \int_{L_1}\nabla_{\dot\theta}W d\varphi + \int_{L_1}\mu_3\frac{\partial \dot Z_3}{\partial\dot\theta}d\varphi = 0$$

$$\nabla_w T - \frac{d}{dt}\nabla_{\dot w}T - r^2 n_{03}w - \int_{L_1}\nabla_{\dot w}W d\varphi = 0 \tag{2.5}$$

$$\nabla_u T - \frac{d}{dt}\nabla_{\dot u}T - n_0 - n_{01}u + n_{11}u'' - (m_{21} - m_{20})\upsilon' - \nabla_{\dot u}W\chi(\varphi) +$$

$$+\lambda(1+u+\upsilon') - [\lambda(u'-\upsilon)]' - r\mu_3\chi(\varphi)\sin\vartheta = 0$$

$$\nabla_\upsilon T - \frac{d}{dt}\nabla_{\dot\upsilon}T - n_{02}\upsilon + n_{12}\upsilon'' - (m_{12} - m_{02})u' - \nabla_{\dot\upsilon}W\chi(\varphi) -$$

$$-[\lambda(1+u+\upsilon')]' - \lambda(u'-\upsilon) - r\mu_3\chi(\varphi)\cos\vartheta = 0$$

$$\rho r^3[\dot u]_k\dot\varphi_k + [n_{11}u' - \lambda(u'-\upsilon)]_k = 0, \quad k = 1, 2$$

$$\rho r^3[\dot\upsilon]_k\dot\varphi_k + [n_{12}\upsilon' - \lambda(1+u+\upsilon')]_k = 0, \quad k = 1, 2$$

Here $[f(\vartheta)]_k = f(\vartheta_k + 0) - f(\vartheta_k - 0)$, $\vartheta_k = \theta + \varphi_k$ is the jump in the function at the final point of the contact area. The function $\chi(\varphi)$ takes the value of unity if $\varphi \in L_1$, and zero if $\varphi \in L_2$. In the last condition, relating the jumps in (2.5), the function

$$v' \cong -u \quad \text{and} \quad \dot{v} \cong -\int_0^{\varphi} \dot{u}\, d\varphi + \dot{v}(t, 0)$$

apart from small first-order terms. Then $[\vartheta']_k = [\dot{v}]_k = 0$ and $[\lambda]_k = 0$, i.e. the tension of the centre line of the tread does not undergo jumps at the boundary points of the contact area. We will represent the penultimate condition, relating the jumps in (2.5), in the form

$$\rho r^3 \dot{\varphi}_k [\dot{u}_k] + (n_{11} - \lambda)[u']_k = 0, \quad k = 1, 2 \tag{2.6}$$

The remaining 10 equations in (2.5), together with the two conditions imposed on the jumps (2.6), and the holonomic constraints (1.6) and (2.3), form a closed system of 14 equations in 14 unknowns: $X_1, X_2, X_3, \beta_0, \beta, \kappa, \theta, w, u, \upsilon, \varphi_1, \varphi_2, \lambda, \mu_3$.

3. THE MOTION OF A LOCKED WHEEL

We will consider the two simplest modes of slipping of a locked wheel. We will assume that the contact area of the wheel with the plane OX_1X_2 moves forward with constant speed, namely, $\dot{X}_1 = V\cos\gamma$, $\dot{X}_2 = V\sin\gamma$, X_3, β, β_0, κ, θ, w, $u(\varphi)$, $\upsilon(\varphi)$, φ_1, φ_2, $\lambda(\varphi)$, $\mu_3(\varphi)$ are constant quantities. Equations (2.5) in the case considered becomes essentially the conditions of equilibrium of a mechanical system with respect to a system of coordinates connected with the disc of the wheel and which translates with constant speed. We obtain these conditions from (2.5) by substituting into them functions corresponding to the steady mode. Note that

$$\nabla_{\dot{q}_i} W = 2fplr\frac{\partial \dot{Z}}{\partial \dot{q}_i} = \frac{2fplr}{V}\left[(\dot{X}_1\cos\beta_0 + \dot{X}_2\sin\beta_0)\frac{\partial \dot{Z}_1}{\partial \dot{q}_i} + (-\dot{X}_1\sin\beta_0 + \dot{X}_2\cos\beta_0)\frac{\partial \dot{Z}_2}{\partial \dot{q}_i}\right]$$

As a result we obtain the following system of equations

$$\begin{aligned}
&-2fplrV^{-1}\dot{X}_1(\varphi_2 - \varphi_1) + F_1\cos\beta - F_2\sin\beta = 0\\
&-2fplrV^{-1}\dot{X}_2(\varphi_2 - \varphi_1) + F_1\sin\beta + F_2\cos\beta = 0\\
&P = \int_{L_1}\mu_3(\varphi)d\varphi, \quad M_2\sin\kappa + M_3\cos\kappa = n_{05}\Delta\beta = 0, \quad M_1 = n_{05}\kappa\\
&M_2 + 2fplrV^{-1}X_3(\dot{X}_1\cos\beta_0 + \dot{X}_2\sin\beta_0)(\varphi_2 - \varphi_1) - \int_{L_1}\mu_3 r\cos\vartheta\, d\varphi = 0\\
&n_{03}w + 2fplV^{-1}(-\dot{X}_1\sin\beta_0 + \dot{X}_2\cos\beta_0)(\varphi_2 - \varphi_1) = 0\\
&N_1(u, \upsilon) - 2fplr^2V^{-1}(\dot{X}_1\cos\beta_0 + \dot{X}_2\sin\beta_0)\chi(\varphi)\cos\vartheta + M_+(u, \upsilon) - r\mu_3\chi(\varphi)\sin\vartheta = 0\\
&N_2(u, \upsilon) + 2fplr^2V^{-1}(\dot{X}_1\cos\beta_0 + \dot{X}_2\sin\beta_0)\chi(\varphi)\sin\vartheta - M_-(u, \upsilon) - r\mu_3\chi(\varphi)\cos\vartheta = 0\\
&(n_{11} - \lambda)[u']_k = 0, \quad k = 1, 2
\end{aligned} \tag{3.1}$$

Here

$$N_1(u, \upsilon) = -n_0 - n_{01}u + n_{11}u'' - (m_{21} - m_{20})\upsilon', \quad M_+(u, \upsilon) = \lambda(1 + u + \upsilon') - [\lambda(u' - \upsilon)]'$$

$$N_2(u, \upsilon) = -n_{02}\upsilon + n_{12}\upsilon'' - (m_{12} - m_{02})u', \quad M_-(u, \upsilon) = [\lambda(1 + u + \upsilon)]' + \lambda(u' - \upsilon)$$

It follows from relations (3.1) that the angle $\beta = \beta_0$. Without loss of generality, we can take $\beta = 0$. The first two relations of system (3.1) can then be represented in the form

$$F_1 = 2fplr(\varphi_2 - \varphi_1)\cos\gamma, \quad F_2 = 2fplr(\varphi_2 - \varphi_1)\sin\gamma$$

Further

$$M_1 = n_{05}\kappa, \quad M_3 = -M_2 \mathrm{tg}\kappa, \quad w = -2fpl\sin\gamma n_{03}^{-1}(\varphi_2 - \varphi_1)$$

The equations which define the components of the displacement vector of points on the tread $u(\varphi)$ and $\upsilon(\varphi)$ in the contact area and outside it, take the form:
when $\varphi \in L_1$

$$\begin{aligned} &N_1(u, \upsilon) + 2fplr^2\cos\gamma\sin\varphi + M_+(u, \upsilon) - r\mu_3\cos\varphi = 0 \\ &N_2(u, \upsilon) + 2fplr^2\cos\gamma\cos\varphi - M_-(u, \upsilon) + r\mu_3\sin\varphi = 0 \end{aligned} \tag{3.2}$$

when $\varphi \in L_2$

$$N_1(u, \upsilon) + M_+(u, \upsilon) = 0, \quad N_2(u, \upsilon) - M_-(u, \upsilon) = 0 \tag{3.3}$$

In Eqs (3.2) the angle θ under steady conditions is taken to be $\pi/2$. In Eqs (3.3) we put $\lambda = n_0 + \lambda_1$, where λ_1 is a small quantity, and we linearize the system obtained. We have

$$\begin{aligned} &-n_{01}u + n_{11}u'' - (m_{21} - m_{20})\upsilon' + \lambda_1' - n_0(u' - \upsilon)' = 0 \\ &-n_{02}\upsilon + n_{12}\upsilon'' - (m_{12} - m_{02})u' - \lambda_1' - n_0(u' - \upsilon) = 0, \quad u + \upsilon' = 0 \end{aligned} \tag{3.4}$$

The last equation in system (3.4) is the linearized condition for the centre line of the tread of the tyre to be inextensible (2.3).
We eliminate the unknowns u and λ_1 from Eqs (3.4) and obtain the equation

$$a_0\upsilon^{(4)} + a_1\upsilon'' + a_2\upsilon = 0$$

$$a_0 = n_{11} - n_0 > 0, \quad a_1 = m_{21} + m_{02} - m_{20} - m_{12} - n_{01} - 2n_0 - n_{12}, \quad a_2 = n_{02} - n_0 > 0$$

the general solution of which in the section $L_2 = [\varphi_2, 2\pi + \varphi_1]$ has the form (everywhere henceforth summation is carried out from $k = 1$ to $k = 4$)

$$\upsilon_2(\varphi) = \sum\nolimits_\upsilon(\varphi), \quad \sum\nolimits_\upsilon(\varphi) = \sum C_k \exp(p_k\varphi), \quad p_{1,\dots,4} = \pm\sqrt{\frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_0}} \tag{3.5}$$

where $C_1, \dots, C_4$ are arbitrary constants. The function $u_2(\varphi)$ in the section L_2 is equal to

$$u_2(\varphi) = -\upsilon_2'(\varphi) = -\sum\nolimits_u(\varphi), \quad \sum\nolimits_u(\varphi) = \sum C_k p_k \exp(p_k\varphi) \tag{3.6}$$

In the contact area $L_1 = [\varphi_1, \varphi_2]$ the functions $u_1(\varphi)$, $\upsilon_1(\varphi)$ are defined by relations (1.7), which, taking the equality $\vartheta = \pi/2 + \varphi$ into account, can be written in the form

$$u_1 = -\varphi\sin\varphi + X_3r^{-1}\cos\varphi - 1, \quad \upsilon_1 = -\varphi\cos\varphi - X_3r^{-1}\sin\varphi \tag{3.7}$$

Since the angle φ is small in the contact area, linearization of relations (3.7) leads to the equalities

$$u_1 = X_3r^{-1} - 1, \quad \upsilon_1 = -\varphi(X_3r^{-1} + 1) \tag{3.8}$$

The conditions which relate the jumps at the boundary points of the contact area, have the form

$$u_1(\varphi_1) = u_2(2\pi + \varphi_1), \quad u_1(\varphi_2) = u_2(\varphi_2), \quad \upsilon_1(\varphi_1) = \upsilon_2(2\pi + \varphi_1), \quad \upsilon_1(\varphi_2) = \upsilon_2(\varphi_2)$$

and, taking relations (3.5), (3.6) and (3.8) into account, can be represented in the form

$$\begin{aligned} &X_3r^{-1} - 1 = -\sum\nolimits_u(\varphi_2) \approx -\sum C_k p_k, \quad X_3r^{-1} - 1 = -\sum\nolimits_u(2\pi + \varphi_1) \approx -\sum\nolimits_u(2\pi) \\ &-\varphi_2(X_3r^{-1} + 1) = \sum\nolimits_\upsilon(\varphi_2) \approx \sum C_k, \quad -\varphi_1(X_3r^{-1} + 1) = \sum\nolimits_\upsilon(2\pi + \varphi_1) \approx \sum\nolimits_\upsilon(2\pi) \end{aligned} \tag{3.9}$$

From Eqs (3.9) we determined the coefficients $C_1, \ldots, C_4$ and then, the functions $\upsilon_2(\varphi)$ and $u_2(\varphi)$ in the form

$$\upsilon_2(\varphi) = \sum D_k \exp(p_k\varphi), \quad u_2(\varphi) = -\sum D_k p_k \exp(p_k\varphi)$$
$$D_k = d_{1k}\varphi_1 + d_{2k}\varphi_2 + d_{3k}(1 - X_3 r^{-1})$$

The expressions for the coefficients d_{sk} ($s = 1, 2, 3$; $k = 1, \ldots, 4$) are not given here in view of their complexity. We merely note that, on the left-hand sides of Eqs (3.9) the coefficient $1 + X_3 r^{-1} \approx 2$. From the first relation of (3.4) we obtain

$$\lambda_1 = \upsilon'''(n_{11} - n_0) + \upsilon'(m_{21} - m_{20} - n_0 - n_{01}) =$$
$$= \sum D_k p_k \exp(p_k\varphi)[(n_{11} - n_0)p_k^2 + (m_{21} - m_{20} - n_0 - n_{01})]$$

Hence, we have found the Lagrange multiplier $\lambda = n_0 + \lambda_1$, the value of which determines the tension of the centre line of the tread outside the contact area. The Lagrange multipliers λ and μ_3 in the area of contact of the wheel with the plane are determined from the first two equations of system (3.2), taking expressions (3.7) into account, in the form

$$\begin{aligned} \lambda' &= -2fplr^2\cos\gamma + [n_0 + 3n_{11} - 2(m_{21} - m_{20}) - 2n_{02} - 4n_{12} - 3(m_{12} - m_{02})]\varphi \\ r\mu_3 &= -n_0 - 3n_{11} + 2(m_{21} - m_{20}) + n_{01}(1 - X_3 r^{-1}) \end{aligned} \tag{3.10}$$

In expressions (3.10) we have retained terms of the zeroth and first order of smallness after calculating the derivatives of the functions represented in (3.7). The multiplier μ_3, according to (3.10), is a constant, apart from terms of the second order of smallness, while the tension in the tread in the contact area varies linearly. The force P and the moment M_2, which act on the wheel disc, can be found from system (3.1), apart from small first-order terms, in the form

$$P = r^{-1}[-n_0 - 3n_{11} + 2(m_{21} - m_{20})](\varphi_2 - \varphi_1)$$
$$M_2 = -rP(\varphi_2 + \varphi_1)/2 - 2fplrX_3\cos\gamma(\varphi_2 - \varphi_1)$$

The relations obtained enable us to determine all the characteristic steady modes of slipping of the wheel with a tyre, namely, the relations between the forces, moments and functions describing the deformation of the tyre, and the generalized coordinates characterizing the translational motion of the wheel disc.

The second steady motion of the locked wheel with slipping is defined as spinning with constant angular velocity around the axis CX_3, when

$$X_1 = X_2 = 0, \quad X_3 = \text{const}$$
$$\dot\beta_0 = \dot\beta = \omega = \text{const}, \ \kappa = \text{const}, \ w = 0, \ \theta = \pi/2, \ \varphi_1 = -\varphi_2 = \text{const}, \ \lambda(\varphi), \mu_3(\varphi)$$

In this case of steady motion, the work done by the forces of dry friction making virtual displacements can be represented in the form

$$\delta A_f = -fplr \int_{-\varphi_2}^{\varphi_2}\int_{-1}^{1} [\mathbf{e}_3 \times (-r\varphi\mathbf{e}_1 + l\xi\mathbf{e}_2)] \frac{\delta Z_1\mathbf{e}_1 + \delta Z_2\mathbf{e}_2}{\sqrt{l^2\xi^2 + r^2\varphi^2}} d\varphi d\xi \operatorname{sign}\dot\beta_0$$

$$\delta Z_1 = \delta X_1\cos\beta_0 + \delta X_2\sin\beta_0 - \delta\beta_0 l\xi - X_3\delta\theta - \delta u r\sin\varphi - \delta\upsilon r\cos\varphi$$

$$\delta Z_2 = -\delta X_1\sin\beta_0 + \delta X_2\cos\beta_0 - \delta\beta_0 r\varphi + \delta w r$$

and further

$$\delta A_f = -fplr \int_{-\varphi_2}^{\varphi_2}\int_{-1}^{1} \sqrt{l^2\xi^2 + r^2\varphi^2} d\varphi d\xi \delta\beta_0 \operatorname{sign}\dot\beta_0 \approx -2fplr\varphi_2\sqrt{l^2 + r^2\varphi_2^2}\delta\beta_0 \operatorname{sign}\dot\beta_0 \tag{3.11}$$

The integral in (3.11) is equal to the sum of four integrals, which can be evaluated approximately by replacing the integrand by $\sqrt{l^2 + r^2\varphi_2^2/2}$. In fact, this replacement indicates that the friction force field in the contact area is replaced by four friction forces, applied at points with coordinates $\xi = \pm 1/2$, $\varphi = \pm\varphi_2/2$.

System of equations (2.5) can be transformed into the following system

$$
\begin{gathered}
F(\beta) = 0, \quad F\left(\beta - \frac{\pi}{2}\right) = 0, \quad P = \int_{L_1} \mu_3(\varphi) d\varphi, \quad r^2 n_{03} w = 0 \\
M_2 \sin\kappa + M_3 \cos\kappa - n_{05}\Delta\beta = 0, \quad n_{05}\Delta\beta - 2fplr\varphi_2\sqrt{l^2 + r^2\varphi_2^2}\operatorname{sign}\dot{\beta}_0 = 0 \\
(J_{2d} - J_{1d})\omega^2 \cos\kappa \sin\kappa + M_1 - n_{05}\kappa = 0, \quad M_2 - \int_{L_1} \mu_3 r\varphi d\varphi = 0 \\
\rho r^3\omega^2[(1+u)\sin^2\varphi + \upsilon\sin\varphi\cos\varphi] - n_0 - n_{01}u + n_{11}u'' - (m_{21} - m_{20})\upsilon' + \\
+ \lambda(1 + u + \upsilon') - [\lambda(u' - \upsilon)]' - r\mu_3\chi(\varphi)\cos\varphi = 0 \\
\rho r^3\omega^2[(1+u)\sin\varphi\cos\varphi + \upsilon\cos^2\varphi] - n_{02}\upsilon + n_{12}\upsilon'' - (m_{12} - m_{02})u' - \\
- [\lambda(1 + u + \upsilon')]' - \lambda(u' - \upsilon) + r\mu_3\chi(\varphi)\sin\varphi = 0
\end{gathered}
\tag{3.12}
$$

From Eqs (3.12) we can determine the relations between the parameters of motion and also the forces and moments required to achieve spinning

$$
F_1 = F_2 = 0, \quad M_1 = n_{05}\kappa + (J_{1d} - J_{2d})\omega^2 \sin\kappa\cos\kappa, \quad M_2 = 0
$$

$$
M_3\cos\kappa = n_{05}\Delta\beta, \quad w = 0, \quad \Delta\beta = 2fplrn_{05}^{-1}\varphi_2\sqrt{l^2 + r^2\varphi_2^2}\operatorname{sign}\dot{\beta}_0
$$

Note further that the functions $\mu_3(\varphi)$, $\lambda(\varphi)$, $u(\varphi)$ are even, while the function $\upsilon(\varphi)$ is odd. The approximate equations $u \cong r^{-1}X_3 - 1$, $u' \cong -3\varphi$, $u'' \cong -3$, $\upsilon \cong -2\varphi$, $\upsilon' \cong -2$, $\upsilon'' \cong 4$ follow from formulae (1.7) in the contact area L_1. Using these equations, we obtain from the last two equations of system (3.12)

$$
r\mu_3 = -n_0 + n_{01}(1 - r^{-1}X_3) + 2(m_{21} - m_{20}) - 3n_{11}
$$

$$
\lambda' = [\rho r^3\omega^2 - 2n_{02} - 4n_{12} - 3(m_{12} - m_{02}) + n_0 + 3n_{11} - 2(m_{21} - m_{20})]\varphi
$$

With the accuracy assumed above, we can determine the relation between the value of the contact area and the vertical force

$$
P = 2\varphi_2\mu_3 = 2\varphi_2 r^{-1}[-n_0 + 2(m_{21} - m_{20}) - 3n_{11}]
$$

To determine the form of the deformed tread it is necessary to obtain a solution of the last two equations of system (3.12), taking into account the conditions imposed on the jump at the boundary points of the contact area $[\lambda]_k = 0$, $[\upsilon']_k = [u']_k = 0$. The solution of this problem is in many ways similar to the solution of the problem of determining the form of the tread in the case of the translational motion of a wheel with a constant speed. If we put $u = \upsilon = 0$ in these equations in the terms containing centrifugal forces of inertia (the terms with ω^2), and assume $\lambda = n_0 - \rho r^3\omega^2 + \lambda_1$, then after eliminating u and λ_1 from the system of linear equations obtained, we derive the equation

$$
b_0\upsilon^{(4)} + b_1\upsilon'' + b_2\upsilon = \frac{3}{2}\rho r^3\omega^2\sin 2\varphi
\tag{3.13}
$$

$$b_0 = n_{11} - n_0 + \rho r^3 \omega^2$$

$$b_1 = m_{21} + m_{02} - m_{20} - m_{12} - n_{01} - n_{12} - 2n_0 + 2\rho r^3 \omega^2$$

$$b_2 = n_{02} - n_0 + \rho r^3 \omega^2$$

We obtain a particular solution of Eq. (3.13) in the form

$$\upsilon_p = \frac{3}{2}\rho r^3 \omega^2 (16b_0 - 4b_1 + b_2)^{-1} \sin 2\varphi$$

while the general solution of homogeneous equation (3.13) υ_g has the form (3.5) with modified coefficients p_k and symmetrical boundary conditions when $\varphi_1 = -\varphi_2$. Correspondingly, the modified coefficients $\tilde{C}_1, \dots, \tilde{C}_4$ are found from a system of equations of the type (3.9)

$$1 - X_3 r^{-1} - 3\rho r^3 \omega^2 (16b_0 - 4b_1 + b_2)^{-1} \approx \sum \tilde{C}_k \tilde{p}_k \approx \sum \tilde{C}_k \tilde{p}_k \exp(2\pi \tilde{p}_k)$$

$$-\varphi_2 [2 + 3\rho r^3 \omega^2 (16b_0 - 4b_1 + b_2)^{-1}] \approx \sum \tilde{C}_k \approx -\sum \tilde{C}_k \exp(2\pi \tilde{p}_k)$$

in the form

$$\tilde{C}_k = e_{1k}\varphi_2 + e_{2k}[1 - X_3 r^{-1} - 3\rho r^3 \omega^2 (16b_0 - 4b_1 + b_2)^{-1}], \quad k = 1, \dots, 4$$

with the condition that

$$3\rho r^3 \omega^2 (16b_0 - 4b_1 + b_2)^{-1} \ll 2$$

Expressions for the coefficients e_{sk} ($s = 1, 2; k = 1, \dots, 4$) are not given here in view of their complexity. The shape of the deformed tread outside the contact area is symmetrical about the plane Cy_1y_2 and is described by the functions

$$u_2(\varphi) = -\upsilon_2'(\varphi), \quad \upsilon_2(\varphi) = 3\rho r^3 \omega^2 (16b_0 - 4b_1 + b_2)^{-1} \sin\varphi \cos\varphi + \sum \tilde{C}_k \exp(\tilde{p}_k \varphi)$$

The correction $\lambda_1(\varphi)$ to the tension of the tread outside the contact area is found from the penultimate equation of system (3.12) taking the functions $u_2(\varphi)$ and $\upsilon_2(\varphi)$ obtained above into account. We have thus completed the determination of all the steady-spinning characteristics and the forces and moments required to obtain it.

REFERENCES

1. VIL'KE, V. G. and KOZHEVNIKOV, I. F., The rolling of a wheel with a reinforced tyre on a plane without slipping. *Prikl. Mat. Mekh.*, 2001, **65**, 6, 944–947.
2. ODEN, J. T., *Finite Elements in Nonlinear Continua*. McGraw-Hill, New York, 1972.
3. VIL'KE, V. G., The conditions for a wheel with a reinforced tyre to roll without slipping. *Vestnik MGU. Ser. 1. Mat. Mekh.*, 2002, 5, 38–42.
4. ISHLINSKII, A. Yu., *Mechanics. Ideas, Problems and Applications*. Nauka, Moscow, 1985.
5. DUVAUT, G. and LIONS, J. L., *Les inéquations en mécanique et en physique*. Dunod, Paris, 1972.
6. KALKER, J. J., *Three-dimensional Elastic Bodies in Rolling Contact*. Kluwer, Dordrecht, 1990.
7. ZHURAVLEV, V. F., A model of dry friction in the problem of the rolling of rigid bodies. *Prikl. Mat. Mekh.*, 1998, **62**, 5, 762–767.
8. VIL'KE, V. G. and DVORNIKOV, M. V., The rolling of a wheel with a pneumatic tyre on a plane. *Prikl. Mat. Mekh.*, 1998, **62**, 3, 393–404.

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